Online Supplement: Dynamic Pricing and Inventory Management under Fluctuating Procurement Costs

Proofs of Statements

We use ∂ to denote the derivative operator of a single variable function, ∂_x to denote the partial derivative operator of a multi-variable function with respect to variable x, and $1_{\{\cdot\}}$ to denote the indicator function. For any multivariate continuously differentiable function $f(x_1, x_2, \cdots, x_n)$ and $\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ in $f(\cdot)$'s domain, $\forall i$, we use $\partial_{x_i} f(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ to denote $\partial_{x_i} f(x_1, x_2, \cdots, x_n)|_{x=\tilde{x}}$. The following lemma is used throughout our proof.

LEMMA 2. *Let $F_i(z, Z)$ be a continuously differentiable and jointly concave function in (z, Z) for $i = 1, 2$, where $z \in [\underline{z}, \bar{z}]$ ($\underline{z}$ and $\bar{z}$ might be infinite) and $Z \in \mathbb{R}^n$. For $i = 1, 2$, let $(z_i, Z_i) := \arg\max_{(z,Z)} F_i(z, Z)$ be the optimizers of $F_i(\cdot, \cdot)$. If $z_1 < z_2$, we have: $\partial_z F_1(z_1, Z_1) \leq \partial_z F_2(z_2, Z_2)$.*

Proof: $z_1 < z_2$, so $\underline{z} \leq z_1 < z_2 \leq \bar{z}$. Hence, $\partial_z F_1(z_1, Z_1) \begin{cases} = 0 & \text{if } z_1 > \underline{z}, \\ \leq 0 & \text{if } z_1 = \underline{z}; \end{cases}$ and $\partial_z F_2(z_2, Z_2) \begin{cases} = 0 & \text{if } z_2 < \bar{z}, \\ \geq 0 & \text{if } z_2 = \bar{z}, \end{cases}$ i.e., $\partial_z F_1(z_1, Z_1) \leq 0 \leq \partial_z F_2(z_2, Z_2)$. *Q.E.D.*

Proof of Lemma 1: We prove parts (a) - (c) together, using backward induction.

We first show, by backward induction, that if the normalized value function, $V_{t-1}(I_{t-1}|c_{t-1}) - c_{t-1}I_{t-1}$, is concavely decreasing in I_{t-1} for any c_{t-1}, we have $H_t(y|c_t)$ is concavely decreasing, $J_t(x_t, q_t, d_t|c_t)$ is jointly concave, and $V_t(I_t|c_t) - c_tI_t$ is concavely decreasing for any given c_t. It is clear that $V_0(I_0|c_0) - c_0I_0 = -c_0I_0$ is concavely decreasing for any c_0, so the initial condition is satisfied. Moreover, it's clear from the continuous distribution of ϵ_t that $L(\cdot)$ is continuously differentiable and concavely decreasing.

For any realization of ϵ_t and ξ_t, $h_t(y|\epsilon_t, \xi_t) := \alpha[V_{t-1}(y - \epsilon_t|s_t(c_t, \xi_t)) - s_t(c_t, \xi_t)(y - \epsilon_t)]$ is concavely decreasing in y since $V_{t-1}(I_{t-1}|c_{t-1}) - c_{t-1}I_{t-1}$ is concavely decreasing for any c_{t-1}. Because concavity is preserved under expectation, $H_t(y|c_t) = \mathbb{E}_{\epsilon_t, \xi_t}\{h_t(y|\epsilon_t, \xi_t)\}$ is also concavely decreasing in y for any c_t.

For any fixed c_t, $R(d_t|c_t) = (p(d_t) - b - \alpha\mu_t(c_t))d_t = R(d_t) - (b + \alpha\mu_t(c_t))d_t$ is strictly concave in d_t. $(b - c_t + \alpha\mu_t(c_t))x_t$ and $(\alpha\mu_t(c_t) - \gamma c_t)q_t$ are linear and, thus, concave in x_t and q_t, respectively. Since $L(\cdot)$ and $H_t(\cdot|c_t)$ is concave for any given c_t, $L(x_t - d_t)$ and $H_t(x_t + q_t - d_t|c_t)$ are jointly concave in (x_t, d_t) and (x_t, q_t, d_t), respectively. Therefore,

$$J_t(x_t, q_t, d_t|c_t) = R(d_t|c_t) + (b - c_t + \alpha\mu_t(c_t))x_t + (\alpha\mu_t(c_t) - \gamma c_t)q_t + L(x_t - d_t) + H_t(x_t + q_t - d_t|c_t)$$

is jointly concave in (x_t, q_t, d_t) for any c_t. Concavity is preserved under maximization, so $V_t(I_t|c_t)$ is also concave in I_t. Suppose $I_1 > I_2$, $F(I_1) \subset F(I_2)$, so we have

$$V_t(I_1|c_t) - c_tI_1 = \max_{(x_t,q_t,d_t) \in F(I_1)} J_t(x_t, q_t, d_t|c_t) \leq \max_{(x_t,q_t,d_t) \in F(I_2)} J_t(x_t, q_t, d_t|c_t) = V_t(I_2|c_t) - c_tI_2,$$

i.e., $V_t(I_t|c_t) - c_tI_t$ is concavely decreasing.

Next, we show that if $V_{t-1}(I_{t-1}|c_{t-1})$ is continuously differentiable in I_{t-1} for any c_{t-1}, $J_t(x_t,q_t,d_t|c_t)$ and $V_t(I_t|c_t)$ are continuously differentiable for any c_t. For $t=0$, $V_0(I_0|c_0)=0$ is continuously differentiable for any c_0. To show the continuous differentiability of $J_t(x_t,q_t,d_t|c_t)$ for any c_t, since $R(d_t|c_t)+(b-c_t+\alpha\mu_t(c_t))x_t+(\alpha\mu_t(c_t)-\gamma c_t)q_t+L(x_t-d_t)$ is continuously differentiable in (x_t,q_t,d_t) for any c_t, it suffices to prove that $H_t(y|c_t)$ is continuously differentiable for any c_t. Since ϵ_t is continuous and $V_{t-1}(I_{t-1}|c_{t-1})$ is continuously differentiable for any c_{t-1}, $\mathbb{E}_{\epsilon_t}\{h_t(y|\epsilon_t,\xi_t)|\xi_t\}$ is continuously differentiable in y and its derivative is given by: $\partial_y\mathbb{E}_{\epsilon_t}\{h_t(y|\epsilon_t,\xi_t)|\xi_t\}=\mathbb{E}_{\epsilon_t}\{\alpha[\partial_{I_{t-1}}V_{t-1}(y-\epsilon_t|s_t(c_t,\xi_t))-s_t(c_t,\xi_t)(y-\epsilon_t)]|\xi_t\}$, where the exchange of differentiation and expectation is easily justified using the canonical argument (See, e.g., Theorem A.5.1 of Durrett (2010), the condition of which can be easily checked observing the continuity of $\partial_{I_{t-1}}V_{t-1}(I_{t-1}|s_t(c_t,\xi_t))$ and that the distribution of ϵ_t is continuous.). Apply the same exchangeability of differentiation and expectation argument, we have, given any c_t, $H_t(y|c_t)$ is continuously differentiable and its derivative is given by

$$\partial_y H_t(y|c_t)=\partial_y\mathbb{E}_{\epsilon_t,\xi_t}\{h_t(y|\epsilon_t,\xi_t)\}=\mathbb{E}_{\xi_t}\{\partial_y\mathbb{E}_{\epsilon_t}\{h_t(y|\epsilon_t,\xi_t)|\xi_t\}\}=\mathbb{E}_{\epsilon_t,\xi_t}\{\alpha[\partial_{I_{t-1}}V_{t-1}(y-\epsilon_t|s_t(c_t,\xi_t))-s_t(c_t,\xi_t)(y-\epsilon_t)]\}.$$

Hence, $J_t(x_t,q_t,d_t|c_t)$ is concave and continuously differentiable for any c_t. By the envelope theorem, $V_t(I_t|c_t)=c_tI_t+\max_{(x_t,q_t,d_t)\in F(I_t)}J_t(x_t,q_t,d_t|c_t)$ is continuously differentiable in I_t.

It remains to show the finiteness of $V_t(I_t|c_t)$. Note that $V_t(I_t|c_t)\le(\sum_{i=1}^{t}\alpha^{i-1})\bar{p}\bar{d}$ and is, thus, uniformly bounded from above by $(\sum_{t=1}^{T}\alpha^{t-1})\bar{p}\bar{d}$. Hence, all statements in Lemma 1 hold. *Q.E.D.*

Proof of Theorem 1: Parts (a) - (b) follow directly from the joint concavity of $J_t(\cdot,\cdot,\cdot|c_t)$.

Now we show **part (d)**. The continuity of $x_t^*(I_t,c_t)$, $q_t^*(I_t,c_t)$, and $d_t^*(I_t,c_t)$ follows from the concavity of $J_t(\cdot,\cdot,\cdot|c_t)$. For the monotonicity results, we only need to consider the case $I_t\ge x_t(c_t)$, i.e., $x_t^*(I_t,c_t)=I_t$. First, we show $x_t^*(I_t,c_t)+q_t^*(I_t,c_t)$ and $d_t^*(I_t,c_t)$ are increasing in I_t. Let $w_t:=I_t+q_t$, we rewrite the objective function for the case $I_t\ge x_t(c_t)$ as

$$J_t^1(w_t,d_t,I_t|c_t)=R(d_t)+\Lambda(I_t-d_t)+\Psi_t(w_t-d_t|c_t)-\gamma c_tw_t+\gamma c_tI_t,$$

where $\Lambda(\cdot)$ and $\Psi_t(\cdot|\cdot)$ are defined in (2). Since $\Lambda(\cdot)$ and $\Psi_t(\cdot|c_t)$ are concave in y for each fixed c_t, $J_t^1(\cdot,\cdot,\cdot|c_t)$ is jointly supermodular in (w_t,d_t,I_t). Since the feasible set $[I_t,+\infty)\times[\underline{d},\bar{d}]\times\mathbb{R}$ is a lattice, $x_t^*(I_t,c_t)+q_t^*(I_t,c_t)=w_t^*(I_t,c_t)$ and $d_t^*(I_t,c_t)$ are increasing in I_t for any fixed c_t.

Next, we show $\Delta_t^*(I_t,c_t)$ is increasing, whereas $q_t^*(I_t,c_t)$ is decreasing, in I_t. Rewrite the objective function as

$$J_t^2(\Delta_t,-q_t,I_t|c_t)=R(I_t-\Delta_t)+\Lambda(\Delta_t)+\Psi_t(\Delta_t-(-q_t)|c_t)+\gamma(-q_t).$$

Since $\Lambda(\cdot)$ and $\Psi_t(\cdot|c_t)$ are concave in y for each fixed c_t, $J_t^2(\cdot,\cdot,\cdot|c_t)$ is jointly supermodular in $(\Delta_t,-q_t,I_t)$. Since the feasible set $[I_t-\bar{d},I_t-\underline{d}]\times(-\infty,0]\times\mathbb{R}$ is a lattice, $\Delta_t^*(I_t,c_t)$ and $-q_t^*(I_t,c_t)$ are increasing in I_t for any fixed c_t. Thus, $q_t^*(I_t,c_t)$ is decreasing in I_t.

Finally, we show **part (c)**. It remains to show the existence of $I_t^*(c_t)$. Suppose $\lim_{I_t\to+\infty}q_t^*(I_t,c_t)=q_*>0$. Since $V_{t-1}(\cdot|\cdot)$ is uniformly bounded from above by $\sum_{t=1}^{T}\alpha^{t-1}\bar{p}\bar{d}<+\infty$. Hence,

$\lim_{I_{t-1}\to+\infty}\partial_{I_{t-1}}V_{t-1}(I_{t-1}|c_{t-1})\le 0$ and $\lim_{I_t\to+\infty}\partial_{q_t}J_t(I_t,q_*,d_t^*(I_t,c_t)|c_t)\le -\gamma c_t<0$, which violates the first order condition with respect to q_t. Therefore, $q_*=0$. Hence, $I_t^*(c_t)=\min\{I_t: q_t^*(I_t,c_t)=0\}$. *Q.E.D.*

Proof of Theorem 2: First, we rewrite the objective function $J_t(x_t,q_t,d_t|c_t)$ as in Equation (2), where $\Lambda(y):=\mathbb{E}_{\epsilon_t}\{-h(y-\epsilon_t)^+-b(y-\epsilon_t)^-\}$ and $\Psi_t(y|c_t):=\alpha\mathbb{E}_{\epsilon_t,\xi_t}\{V_{t-1}(y-\epsilon_t|s_t(c_t,\xi_t))|c_t\}$.

Part (a). If $b\le c_t-\alpha\mu_t(c_t)$, $b-c_t+\alpha\mu_t(c_t)\le 0$ and, thus, $J_t(\cdot,q_t,d_t)$ is decreasing in x_t for any (q_t,d_t) and c_t. Since we select the lexicographically smallest optimizer, $x_t(c_t)=-\infty$. Now we suppose $\gamma c_t\le c_t-b<\alpha\mu_t(c_t)$. If $x_t(c_t)>-\infty$, the first order condition with respect to q_t implies that $\partial_y\Psi_t(x_t(c_t)+q_t(c_t)-d_t(c_t)|c_t)\le\gamma c_t$, and, hence, $\partial_{x_t}J_t(x_t(c_t),q_t(c_t),d_t(c_t)|c_t)\le b+\gamma c_t-c_t\le 0$, since $\partial_y\Lambda(y)\le b$. The first order condition with respect to x_t suggests that $b+\gamma c_t-c_t=0$ and $\partial_y\Lambda(x_t(c_t)-d_t(c_t))=b=\partial_y\Lambda(-\infty)$. Therefore, $(x_t(c_t)-\delta,q_t(c_t)+\delta,d_t(c_t))$ is another unconstrained optimizer of $J_t(x_t,q_t,d_t|c_t)$, for any $\delta>0$. This contradicts the assumption that $(x_t(c_t),q_t(c_t),d_t(c_t))$ is the lexicographically smallest optimizer. Hence, $x_t(c_t)=-\infty$, if $b\le\max\{c_t-\gamma c_t,c_t-\alpha\mu_t(c_t)\}$.

Part (b). If $\gamma c_t\ge\alpha\mu_t(c_t)$, by Theorem 1(a), $\sup_y\partial_y\Psi_t(y|c_t)\le\alpha\mu_t(c_t)\le\gamma c_t$. Hence, $\sup_{x_t\in\mathbb{R},q_t\ge 0,d_t\in[\underline{d},\bar{d}]}\{\partial_{q_t}J_t(x_t,q_t,d_t|c_t)\}\le\gamma c_t-\gamma c_t\le 0$. Since we choose the lexicographically smallest optimizer, $q_t(c_t)=0$.

Part (c). For $t=1$, observe that $\lim_{y\to-\infty}\partial_yH_1(y|c_1)=-\alpha\mu_1(c_1)$. If $b\le c_1$, $\sup\{\partial_{x_1}J_1(x_1,q_1,d_1|c_1)\}\le b-c_1+\alpha\mu_1(c_1)-\alpha\mu_1(c_1)\le 0$, for any x_t. Hence, $x_t(c_1)=-\infty$. On the other hand, if $b-c_1>0$, $\partial_{x_1}J_1(x_1,q_1,d_1|c_1)\ge\frac{b-c_1}{2}>0$ as $x_1\to-\infty$, i.e., $x_1(c_1)>-\infty$. *Q.E.D.*

Proof of Theorem 3: Part (a) We show that $V_t(I_t|c_t)$ is convexly decreasing in c_t by backward induction. Observe that $V_0(I_0|c_0)=0$ for any I_0 and is, thus, convexly decreasing in c_0. It suffices to show that if $V_{t-1}(I_{t-1}|c_{t-1})$ is convexly decreasing in c_{t-1}, $V_t(I_t|c_t)$ is convexly decreasing in c_t, given $s_t(c_t,\xi_t)$ is concavely increasing in c_t for any realization of ξ_t.

For any $\hat{c}_t,c_t$, let $\eta\in[0,1]$ and $\bar{c}=\eta\hat{c}_t+(1-\eta)c_t$. For any given x_t, q_t, d_t and realized ϵ_t and ξ_t,

$$\begin{aligned}&\eta V_{t-1}(x_t+q_t-d_t-\epsilon_t|s_t(\hat{c}_t,\xi_t))+(1-\eta)V_{t-1}(x_t+q_t-d_t-\epsilon_t|s_t(c_t,\xi_t))\\ \ge&V_{t-1}(x_t+q_t-d_t-\epsilon_t|\eta s_t(\hat{c}_t,\xi_t)+(1-\eta)s_t(c_t,\xi_t))\\ \ge&V_{t-1}(x_t+q_t-d_t-\epsilon_t|s_t(\bar{c},\xi_t)),\end{aligned}$$

where the first inequality follows from the convexity of $V_{t-1}(I_{t-1}|c_{t-1})$ in c_{t-1}, the second from the concavity of $s_t(c_t,\xi_t)$ in c_t and the monotonicity that $V_{t-1}(I_{t-1}|c_{t-1})$ is decreasing in c_{t-1}. Moreover, since $s_t(c_t,\xi_t)$ is increasing in c_t for any realized ξ_t, $V_{t-1}(x_t-d_t-\epsilon_t|s_t(c_t,\xi_t))$ is convexly decreasing in c_t. Since convexity and monotonicity are preserved under expectation,

$$c_tI_t+J_t(x_t,q_t,d_t|c_t)=R(d_t)-c_t(x_t-I_t)-\gamma c_tq_t-\Lambda(x_t-d_t)+\alpha\mathbb{E}_{\epsilon_t,\xi_t}\{V_{t-1}(x_t+q_t-d_t-\epsilon_t|s_t(c_t,\xi_t))\}$$

is convexly decreasing in c_t, since $x_t\ge I_t$. Convexity and monotonicity are preserved under maximization operated on a family of convexly decreasing functions, so $V_t(I_t|c_t)$ is convexly decreasing in c_t. This completes the proof of **part (a)**.

Part (b). We show **part (b)** by backward induction, i.e., if $\hat{s}_t(c_t,\xi_t)\geq_{cx} s_t(c_t,\xi_t)$ and $\hat{V}_{t-1}(I_{t-1}|c_{t-1})\geq V_{t-1}(I_{t-1}|c_{t-1})$ for all (I_{t-1},c_{t-1}), $\hat{V}_t(I_t|c_t)\geq V_t(I_t|c_t)$ for all (I_t,c_t). Since $\hat{V}_0(I_0|c_0)=V_0(I_0|c_0)=0$ for all (I_0,c_0), the initial condition is satisfied.

$$\begin{aligned}\hat{V}_t(I_t|c_t) &= c_tI_t+\max\{R(d_t)-c_tx_t-\gamma c_tq_t+\Lambda(x_t-d_t)\\ &\quad +\alpha\mathbb{E}[\hat{V}_{t-1}(x_t+q_t-d_t-\epsilon_t|\hat{s}_t(c_t,\xi_t))|c_t]:(x_t,q_t,d_t)\in F(I_t)\}\\ &\geq c_tI_t+\max\{R(d_t)-c_tx_t-\gamma c_tq_t+\Lambda(x_t-d_t)\\ &\quad +\alpha\mathbb{E}[\hat{V}_{t-1}(x_t+q_t-d_t-\epsilon_t|s_t(c_t,\xi_t))|c_t]:(x_t,q_t,d_t)\in F(I_t)\}\\ &\geq c_tI_t+\max\{R(d_t)-c_tx_t-\gamma c_tq_t+\Lambda(x_t-d_t)\\ &\quad +\alpha\mathbb{E}[V_{t-1}(x_t+q_t-d_t-\epsilon_t|s_t(c_t,\xi_t))|c_t]:(x_t,q_t,d_t)\in F(I_t)\}\\ &= V_t(I_t|c_t),\end{aligned}$$

where the first inequality follows from the convexity of $\hat{V}_{t-1}(I_{t-1}|\cdot)$, and the second from the inequality $\hat{V}_{t-1}(I_{t-1}|c_{t-1})\geq V_{t-1}(I_{t-1}|c_{t-1})$ for all (I_{t-1},c_{t-1}). *Q.E.D.*

Proof of Theorem 4: First, **part (a)**. As in Equation (3), we rewrite $J_t(x_t,q_t,d_t|c_t)=\tilde{J}_t(\Delta_t,q_t,d_t|c_t)$ in terms of (Δ_t,q_t,d_t). It's clear that maximizing $J_t(x_t,q_t,d_t|c_t)$ is equivalent to maximizing $\tilde{J}_t(\Delta_t,q_t,d_t|c_t)$. By (3), $d_t(c_t)=\arg\max_{d_t\in[\underline{d},\bar{d}]}\{R(d_t)-c_td_t\}$ follows immediately.

We now prove **parts (b) - (c)** together by backward induction.

We need to show that, if $\partial_{I_{t-1}}V_{t-1}(I_{t-1}|\hat{c}_{t-1})\geq\partial_{I_{t-1}}V_{t-1}(I_{t-1}|c_{t-1})$ for any $\hat{c}_{t-1}>c_{t-1}$ and $I_{t-1}\in\mathbb{R}$, for any $\hat{c}_t>c_t$, (a) $d_t(\hat{c}_t)\leq d_t(c_t)$, (b) $d_t^*(I_t,\hat{c}_t)\leq d_t^*(I_t,c_t)$, and (c) $\partial_{I_t}V_t(I_t|\hat{c}_t)\geq\partial_{I_t}V_t(I_t|c_t)$ for all I_t. For $t=0$, $\partial_{I_0}V_0(I_0|\hat{c}_0)=\partial_{I_0}V_0(I_0|c_0)=0$ for any $\hat{c}_t>c_t$. The initial condition is, thus, satisfied.

Without loss of generality, we assume that $x_t(\hat{c}_t)$ and $x_t(c_t)$ are finite, i.e., $x_t(\hat{c}_t),x_t(c_t)>-\infty$. Our argument can be easily extended to the extreme case in which $x_t(\hat{c}_t)=-\infty$ or $x_t(c_t)=-\infty$. We rewrite the objective function $J_t(x_t,q_t,d_t|c_t)$ as (2). First, we show that if $\hat{c}_t>c_t$, $\partial_y\Psi_t(y|\hat{c}_t)\geq\partial_y\Psi_t(y|c_t)$. Since $\hat{c}_t>c_t$, $s_t(\hat{c}_t,\xi_t)\geq_{s.d.}s_t(c_t,\xi_t)$. As in the proof of Lemma 1, we have the following: $\partial_y\Psi_t(y|\hat{c}_t)=\alpha\mathbb{E}_{\epsilon_t,\xi_t}\{\partial_{I_{t-1}}V_{t-1}(y-\epsilon_t|s_t(\hat{c}_t,\xi_t))\}\geq\alpha\mathbb{E}_{\epsilon_t,\xi_t}\{\partial_{I_{t-1}}V_{t-1}(y-\epsilon_t|s_t(c_t,\xi_t))\}=\partial_y\Psi_t(y|c_t)$, where the inequality follows from the assumption that $\partial_{I_{t-1}}V_{t-1}(I_{t-1}|\hat{c}_{t-1})\geq\partial_{I_{t-1}}V_{t-1}(I_{t-1}|c_{t-1})$ for any $\hat{c}_{t-1}>c_{t-1}$.

$d_t(\hat{c}_t)\leq d_t(c_t)$ follows directly from (1) and the concavity of $R(\cdot)$. Now we show $d_t^*(I_t,\hat{c}_t)\leq d_t^*(I_t,c_t)$ for all I_t. If $I_t\leq\min\{x_t(\hat{c}_t),x_t(c_t)\}$, $d_t^*(I_t,\hat{c}_t)=d_t(\hat{c}_t)\leq d_t(c_t)=d_t^*(I_t,c_t)$.

If $I_t\geq\max\{x_t(\hat{c}_t),x_t(c_t)\}$ and $d_t^*(I_t,\hat{c}_t)>d_t^*(I_t,c_t)$, the concavity of $\Lambda(\cdot)$ implies that $\partial_y\Lambda(I_t-d_t^*(I_t,\hat{c}_t))\geq\partial_y\Lambda(I_t-d_t^*(I_t,c_t))$. If $I_t\in[x_t(c_t),I_t^*(c_t)]\cap[x_t(\hat{c}_t),I_t^*(\hat{c}_t)]$, the first order condition with respect to q_t yields that $\partial_y\Psi_t(I_t+q_t^*(I_t,\hat{c}_t)-d_t^*(I_t,\hat{c}_t)|\hat{c}_t)=\gamma\hat{c}_t>\gamma c_t=\partial_y\Psi_t(I_t+q_t^*(I_t,c_t)-d_t^*(I_t,c_t)|c_t)$. If $I_t\geq\max\{I_t^*(\hat{c}_t),I_t^*(c_t)\}$, since $\partial_y\Psi_t(y|\hat{c}_t)\geq\partial_y\Psi_t(y|c_t)$, $\partial_y\Psi_t(I_t-d_t^*(I_t,\hat{c}_t)|\hat{c}_t)\geq\partial_y\Psi_t(I_t-d_t^*(I_t,c_t)|c_t)$. Therefore, if $I_t\in[x_t(c_t),I_t^*(c_t)]\cap[x_t(\hat{c}_t),I_t^*(\hat{c}_t)]$ or $I_t\geq\max\{I_t^*(\hat{c}_t),I_t^*(c_t)\}$, $\partial_y\Psi_t(I_t+q_t^*(I_t,\hat{c}_t)-d_t^*(I_t,\hat{c}_t)|\hat{c}_t)\geq\partial_y\Psi_t(I_t+q_t^*(I_t,c_t)-d_t^*(I_t,c_t)|c_t)$. Since $d_t^*(I_t,\hat{c}_t)>d_t^*(I_t,c_t)$, Lemma 2 yields that $\partial_{d_t}J_t(I_t,q_t^*(I_t,\hat{c}_t),d_t^*(I_t,\hat{c}_t)|\hat{c}_t)\geq\partial_{d_t}J_t(I_t,q_t^*(I_t,c_t),d_t^*(I_t,c_t)|c_t)$. Therefore,

$$\begin{aligned}R'(d_t^*(I_t,\hat{c}_t)) &= \partial_{d_t}J_t(I_t,q_t^*(I_t,\hat{c}_t),d_t^*(I_t,\hat{c}_t)|\hat{c}_t)+\partial_y\Lambda(I_t-d_t^*(I_t,\hat{c}_t))+\partial_y\Psi_t(I_t+q_t^*(I_t,\hat{c}_t)-d_t^*(I_t,\hat{c}_t)|\hat{c}_t)\\ &\geq \partial_{d_t}J_t(I_t,q_t^*(I_t,c_t),d_t^*(I_t,c_t)|c_t)+\partial_y\Lambda(I_t-d_t^*(I_t,c_t))+\partial_y\Psi_t(I_t+q_t^*(I_t,c_t)-d_t^*(I_t,c_t)|c_t)\\ &= R'(d_t^*(I_t,c_t)).\end{aligned}$$

which violates the strict concavity of $R(\cdot)$. This contradiction proves that $d_t^*(I_t,\hat{c}_t) \le d_t^*(I_t,c_t)$ if $I_t \in [x_t(c_t), I_t^*(c_t)] \cap [x_t(\hat{c}_t), I_t^*(\hat{c}_t)]$ or $I_t \ge \max\{I_t^*(\hat{c}_t), I_t^*(c_t)\}$.

If $x_t(\hat{c}_t) \ge x_t(c_t)$ and $I_t \in [x_t(c_t), x_t(\hat{c}_t)]$, $d_t^*(I_t,\hat{c}_t) = d_t^*(x_t(\hat{c}_t),\hat{c}_t) = d_t(\hat{c}_t) \le d_t(c_t) = d_t^*(x_t(c_t),c_t) \le d_t^*(I_t,c_t)$, where the inequalities follow from Theorem 1(d). Likewise, if $x_t(c_t) \ge x_t(\hat{c}_t)$ and $I_t \in [x_t(\hat{c}_t), x_t(c_t)]$, we have $d_t^*(I_t,\hat{c}_t) \le d_t^*(x_t(c_t),\hat{c}_t) \le d_t^*(x_t(c_t),c_t) = d_t^*(I_t,c_t)$. Applying the same argument, we can show that $d_t^*(I_t,\hat{c}_t) \le d_t^*(I_t,c_t)$ for $I_t \in [I_t^*(c_t), I_t^*(\hat{c}_t)]$ or $I_t \in [I_t^*(\hat{c}_t), I_t^*(c_t)]$. Therefore, $d_t^*(I_t,\hat{c}_t) \le d_t^*(I_t,c_t)$ for all I_t.

To complete the induction, we need to show $\partial_{I_t} V_t(I_t|\hat{c}_t) \ge \partial_{I_t} V_t(I_t|c_t)$ for any I_t. First we assume that $I_t \ge \max\{x_t(c_t), x_t(\hat{c}_t)\}$, $\partial_{I_t} V_t(I_t|c_t) = \partial_y \Lambda(I_t - d_t^*(I_t,c_t)) + \partial_y \Psi_t(I_t + q_t^*(I_t,c_t) - d_t^*(I_t,c_t)|c_t)$ and $\partial_{I_t} V_t(I_t|\hat{c}_t) = \partial_y \Lambda(I_t - d_t^*(I_t,\hat{c}_t)) + \partial_y \Psi_t(I_t + q_t^*(I_t,\hat{c}_t) - d_t^*(I_t,\hat{c}_t)|\hat{c}_t)$. If $d_t^*(I_t,\hat{c}_t) < d_t^*(I_t,c_t)$, Lemma 2 implies that $\partial_{d_t} J_t(I_t, q_t^*(I_t,\hat{c}_t), d_t^*(I_t,\hat{c}_t)|\hat{c}_t) \le \partial_{d_t} J_t(I_t, q_t^*(I_t,c_t), d_t^*(I_t,c_t)|c_t)$ and the strict concavity of $R(\cdot)$ implies that $R'(d_t^*(I_t,\hat{c}_t)) > R'(d_t^*(I_t,c_t))$. Therefore,

$$\begin{aligned}
\partial_{I_t} V_t(I_t|\hat{c}_t) &= \partial_y \Lambda(I_t - d_t^*(I_t,\hat{c}_t)) + \partial_y \Psi_t(I_t + q_t^*(I_t,\hat{c}_t) - d_t^*(I_t,\hat{c}_t)|\hat{c}_t) \\
&= R'(d_t^*(I_t,\hat{c}_t)) - \partial_{d_t} J_t(I_t, q_t^*(I_t,\hat{c}_t), d_t^*(I_t,\hat{c}_t)|\hat{c}_t) \\
&> R'(d_t^*(I_t,c_t)) - \partial_{d_t} J_t(I_t, q_t^*(I_t,c_t), d_t^*(I_t,c_t)|c_t) \\
&= \partial_y \Lambda(I_t - d_t^*(I_t,c_t)) + \partial_y \Psi_t(I_t + q_t^*(I_t,c_t) - d_t^*(I_t,c_t)|c_t) \\
&= \partial_{I_t} V_t(I_t|c_t).
\end{aligned}$$

If $d_t^*(I_t,\hat{c}_t) = d_t^*(I_t,c_t)$, there are two cases (a) $q_t^*(I_t,\hat{c}_t) > q_t^*(I_t,c_t)$, and (b) $q_t^*(I_t,\hat{c}_t) \le q_t^*(I_t,c_t)$. If $q_t^*(I_t,\hat{c}_t) > q_t^*(I_t,c_t)$, Lemma 2 implies that $\partial_y \Psi_t(I_t + q_t^*(I_t,\hat{c}_t) - d_t^*(I_t,\hat{c}_t)|\hat{c}_t) - \gamma\hat{c}_t \ge \partial_y \Psi_t(I_t + q_t^*(I_t,c_t) - d_t^*(I_t,c_t)|c_t) - \gamma c_t$, i.e., $\partial_y \Psi_t(I_t + q_t^*(I_t,\hat{c}_t) - d_t^*(I_t,\hat{c}_t)|\hat{c}_t) > \partial_y \Psi_t(I_t + q_t^*(I_t,c_t) - d_t^*(I_t,c_t)|c_t)$. If $q_t^*(I_t,\hat{c}_t) \le q_t^*(I_t,c_t)$, $\partial_y \Psi_t(y|\hat{c}_t) \ge \partial_y \Psi_t(y|c_t)$ implies that $\partial_y \Psi_t(I_t + q_t^*(I_t,\hat{c}_t) - d_t^*(I_t,\hat{c}_t)|\hat{c}_t) \ge \partial_y \Psi_t(I_t + q_t^*(I_t,c_t) - d_t^*(I_t,c_t)|c_t)$. Moreover, since $d_t^*(I_t,\hat{c}_t) = d_t^*(I_t,c_t)$, $\partial_y \Lambda(I_t - d_t^*(I_t,\hat{c}_t)) = \partial_y \Lambda(I_t - d_t^*(I_t,c_t))$. Therefore, $\partial_{I_t} V_t(I_t|\hat{c}_t) \ge \partial_{I_t} V_t(I_t|c_t)$ for all $I_t \ge \max\{x_t(c_t), x_t(\hat{c}_t)\}$.

If $I_t \le \min\{x_t(c_t), x_t(\hat{c}_t)\}$, $\partial_{I_t} V_t(I_t|\hat{c}_t) = \hat{c}_t > c_t = \partial_{I_t} V_t(I_t|c_t)$. If $I_t \in [x_t(c_t), x_t(\hat{c}_t)]$, $\partial_{I_t} V_t(I_t|\hat{c}_t) = \hat{c}_t > c_t \ge \partial_{I_t} V_t(I_t|c_t)$. If $I_t \in [x_t(\hat{c}_t), x_t(c_t)]$, $\partial_{I_t} V_t(I_t|\hat{c}_t) \ge \partial_{I_t} V_t(x_t(c_t)|\hat{c}_t) \ge \partial_{I_t} V_t(x_t(c_t)|c_t) = c_t = \partial_{I_t} V_t(I_t|c_t)$. This completes the induction and, thus, the proof. *Q.E.D.*

Proof of Theorem 5: For all parts, without loss of generality, we assume that $x_t(\hat{c}_t), x_t(c_t) > -\infty$. Our argument can be easily extended to the extreme case in which $x_t(\hat{c}_t) = -\infty$ or $x_t(c_t) = -\infty$.

Part (a). Since $q_t(c_t) > 0$, the first-order condition with respect to q_t implies that $\partial_y \Psi_t(\Delta_t(c_t) + q_t(c_t)|c_t) = \gamma c_t$. Since $x_t(c_t) > -\infty$, the optimality of $\Delta_t(c_t)$ yields that $\partial_{\Delta_t} \tilde{J}_t(\Delta_t(c_t), q_t(c_t), d_t(c_t)) = 0$. Hence, $\partial_y \Lambda(\Delta_t(c_t)) - (1-\gamma)c_t + \partial_y \Psi_t(\Delta_t(c_t) + q_t(c_t)|c_t) - \gamma c_t = 0$ and, thus, $\partial_y \Lambda(\Delta_t(c_t)) - (1-\gamma)c_t = 0$. Since $\Lambda(\cdot)$ is concave, the KKT theorem implies that $\Delta_t(c_t) = \arg\max_{\Delta_t}\{\Lambda(\Delta_t) - (1-\gamma)c_t\Delta_t\}$.

Part (b). We consider the case $\gamma < 1$ only, because the case $\gamma = 1$ follows from similar argument. Since $q_t(c_t) > 0$, the first-order condition with respect to q_t implies that

$$\partial_y \Psi_t(\Delta_t(c_t) + q_t(c_t)|c_t) - \gamma c_t = 0 \ge \partial_y \Psi_t(\Delta_t(\hat{c}_t) + q_t(\hat{c}_t)|\hat{c}_t) - \gamma\hat{c}_t. \tag{4}$$

If $\Delta_t(\hat{c}_t) > \Delta_t(c_t)$, Lemma 2 implies that

$$\partial_y \Lambda(\Delta_t(\hat{c}_t)) + \partial_y \Psi_t(\Delta_t(\hat{c}_t) + q_t(\hat{c}_t)|\hat{c}_t) - \hat{c}_t \geq \partial_y \Lambda(\Delta_t(c_t)) + \partial_y \Psi_t(\Delta_t(c_t) + q_t(c_t)|c_t) - c_t. \tag{5}$$

Inequalities (4) and (5) imply that $\partial_y \Lambda(\Delta_t(\hat{c}_t)) - \partial_y \Lambda(\Delta_t(c_t)) \geq (1-\gamma)(\hat{c}_t - c_t) > 0$, which contradicts the concavity of $\Lambda(\cdot)$. Therefore, $\Delta_t(\hat{c}_t) \leq \Delta_t(c_t)$ and, thus, $x_t(\hat{c}_t) = \Delta_t(\hat{c}_t) + d_t(\hat{c}_t) \leq \Delta_t(c_t) + d_t(c_t) = x_t(c_t)$. $x_t^*(I_t, \hat{c}_t) \leq x_t^*(I_t, c_t)$ follows immediately from $x_t(\hat{c}_t) \leq x_t(c_t)$.

Part (c). Since $q_t(\hat{c}_t) > 0$, the first-order condition with respect to q_t implies that

$$\partial_y \Psi_t(\Delta_t(\hat{c}_t) + q_t(\hat{c}_t)|\hat{c}_t) - \gamma\hat{c}_t = 0 \geq \partial_y \Psi_t(\Delta_t(c_t) + q_t(c_t)|c_t) - \gamma c_t. \tag{6}$$

If $\Delta_t(c_t) > \Delta_t(\hat{c}_t)$, Lemma 2 implies that

$$\partial_y \Lambda(\Delta_t(c_t)) + \partial_y \Psi_t(\Delta_t(c_t) + q_t(c_t)|c_t) - c_t \geq \partial_y \Lambda(\Delta_t(\hat{c}_t)) + \partial_y \Psi_t(\Delta_t(\hat{c}_t) + q_t(\hat{c}_t)|\hat{c}_t) - \hat{c}_t. \tag{7}$$

Inequalities (6) and (7) imply that $\partial_y \Lambda(\Delta_t(c_t)) - \partial_y \Lambda(\Delta_t(\hat{c}_t)) \geq (\gamma - 1)(\hat{c}_t - c_t) > 0$, which contradicts the concavity of $\Lambda(\cdot)$. Therefore, $\Delta_t(\hat{c}_t) \geq \Delta_t(c_t)$, if $\gamma > 1$ and $q_t(\hat{c}_t) > 0$. *Q.E.D.*

Proof of Theorem 6: We prove **parts (a) - (c)** together by backward induction.

We need to show that: under the condition that $\gamma = 1$ and $\kappa_t(c_t)$ is decreasing in c_t, if $\partial_{I_{t-1}} V_{t-1}(I_{t-1}|\hat{c}_{t-1}) - \hat{c}_{t-1} \leq \partial_{I_{t-1}} V_{t-1}(I_{t-1}|c_{t-1}) - c_{t-1}$ for any $\hat{c}_{t-1} > c_{t-1}$, (a) $\Delta_t(\hat{c}_t) \leq \Delta_t(c_t)$, (b) $x_t(\hat{c}_t) \leq x_t(c_t)$, (c) $I_t^*(\hat{c}_t) \leq I_t^*(c_t)$, (d) $q_t^*(I_t, \hat{c}_t) \leq q_t^*(I_t, c_t)$, and (e) $\partial_{I_t} V_t(I_t|\hat{c}_t) - \hat{c}_t \leq \partial_{I_t} V_t(I_t|c_t) - c_t$ for any $\hat{c}_t > c_t$. Note that $\partial_{I_0} V_0(I_0|\hat{c}_0) - \hat{c}_0 = -\hat{c}_0 < -c_0 = \partial_{I_0} V_0(I_0|c_0) - c_0$. Hence, the initial condition is satisfied.

Without loss of generality, we assume that $x_t(\hat{c}_t), x_t(c_t) > -\infty$. Our argument can be easily extended to the extreme case in which $x_t(\hat{c}_t) = -\infty$ or $x_t(c_t) = -\infty$. Since $\gamma = 1$ and $\kappa_t(c_t)$ is decreasing in c_t, $b - c_t + \alpha\mu_t(c_t)$ and the risk-premium of the forward-buying contract $\phi_t(c_t) := \alpha\mu_t(c_t) - \gamma c_t$ are both decreasing in c_t. It's clear that $\partial_{I_{t-1}} V_{t-1}(I_{t-1}|\hat{c}_{t-1}) - \hat{c}_{t-1} \leq \partial_{I_{t-1}} V_{t-1}(I_{t-1}|c_{t-1}) - c_{t-1}$ for any $\hat{c}_{t-1} > c_{t-1}$ implies that $\partial_y H_t(y|\hat{c}_t) \leq \partial_y H_t(y|c_t)$ for any $\hat{c}_t > c_t$. We also have that $\partial_{d_t} R(d_t|\hat{c}_t) \leq \partial_{d_t} R(d_t|c_t)$. The first order condition with respect to x_t implies that:

$$\begin{cases} b - \hat{c}_t + \alpha\mu_t(\hat{c}_t) + \partial_y L(\Delta_t(\hat{c}_t)) + \partial_y H_t(\Delta_t(\hat{c}_t) + q_t(\hat{c}_t)|\hat{c}_t) = 0, \\ b - c_t + \alpha\mu_t(c_t) + \partial_y L(\Delta_t(c_t)) + \partial_y H_t(\Delta_t(c_t) + q_t(c_t)|c_t) = 0. \end{cases} \tag{8}$$

If $q_t(\hat{c}_t) > q_t(c_t)$, Lemma 2 implies that:

$$\begin{aligned} \alpha\mu_t(\hat{c}_t) - \hat{c}_t + \partial_y H_t(\Delta_t(\hat{c}_t) + q_t(\hat{c}_t)|\hat{c}_t) &= \partial_{q_t} J_t(x_t(\hat{c}_t), q_t(\hat{c}_t), d_t(\hat{c}_t)|\hat{c}_t) \\ &\geq \partial_{q_t} J_t(x_t(c_t), q_t(c_t), d_t(c_t)|c_t) \\ &= \alpha\mu_t(c_t) - c_t + \partial_y H_t(\Delta_t(c_t) + q_t(c_t)|c_t) \end{aligned} \tag{9}$$

$$\text{Hence,} \quad b + \partial_y L(\Delta_t(\hat{c}_t)) \leq b + \partial_y L(\Delta_t(c_t)). \tag{10}$$

Since $\alpha\mu_t(\hat{c}_t) - \hat{c}_t \leq \alpha\mu_t(c_t) - c_t$, (9) also implies that $\partial_y H_t(\Delta_t(c_t) + q_t(c_t)|c_t) \leq \partial_y H_t(\Delta_t(\hat{c}_t) + q_t(\hat{c}_t)|\hat{c}_t)$. Hence, $\Delta_t(c_t) + q_t(c_t) \geq \Delta_t(\hat{c}_t) + q_t(\hat{c}_t)$, by $\partial_y H_t(y|\hat{c}_t) \leq \partial_y H_t(y|c_t)$. So we have $\Delta_t(c_t) > \Delta_t(\hat{c}_t)$. Therefore, inequality in (10) must hold as equality. The concavity of $L(\cdot)$ implies that $\partial_y L(\Delta)$ is a constant for $\Delta \in$

$[\Delta_t(\hat{c}_t), \Delta_t(c_t)]$. Since the lexicographically smallest optimizer is selected, the firm with c_t should decrease $\Delta_t(c_t)$ and increase $q_t(c_t)$, which contradicts the selection of $(x_t(c_t), q_t(c_t), d_t(c_t))$. Therefore, $q_t(\hat{c}_t) \le q_t(c_t)$.

If $q_t(\hat{c}_t) = q_t(c_t)$, (8) yields that $\Delta_t(\hat{c}_t) \le \Delta_t(c_t)$ and, hence, $x_t(\hat{c}_t) = d_t(\hat{c}_t) + \Delta_t(\hat{c}_t) \le d_t(c_t) + \Delta_t(c_t) = x_t(c_t)$. If $q_t(\hat{c}_t) < q_t(c_t)$, Lemma 2 implies that the inequalities in (9) and (10) are reversed. The concavity of $L(\cdot)$ yields that $\Delta_t(\hat{c}_t) \le \Delta_t(c_t)$ and, thus, $x_t(\hat{c}_t) \le x_t(c_t)$.

We next show that $q_t^*(I_t, \hat{c}_t) \le q_t^*(I_t, c_t)$ for all I_t. If $I_t \in [x_t(\hat{c}_t), x_t(c_t)]$, $q_t^*(I_t, \hat{c}_t) \le q_t(\hat{c}_t) \le q_t(c_t)$. Assume that $I_t \ge x_t(c_t)$ and $q_t^*(I_t, \hat{c}_t), q_t^*(I_t, c_t) > 0$. The first order condition with respect to q_t suggests that:

$$\begin{cases} \alpha\mu_t(\hat{c}_t) - \hat{c}_t + \partial_y H_t(I_t + q_t^*(I_t, \hat{c}_t) - d_t^*(I_t, \hat{c}_t)|\hat{c}_t) = 0, \\ \alpha\mu_t(c_t) - c_t + \partial_y H_t(I_t + q_t^*(I_t, c_t) - d_t^*(I_t, c_t)|c_t) = 0. \end{cases} \tag{11}$$

Since $\alpha\mu_t(\hat{c}_t) - \hat{c}_t \le \alpha\mu_t(c_t) - c_t$, $\partial_y H_t(I_t + q_t^*(I_t, \hat{c}_t) - d_t^*(I_t, \hat{c}_t)|\hat{c}_t) \ge \partial_y H_t(I_t + q_t^*(I_t, c_t) - d_t^*(I_t, c_t)|c_t)$. Since $\partial_y H_t(y|\hat{c}_t) \le \partial_y H_t(y|c_t)$, $q_t^*(I_t, \hat{c}_t) - d_t^*(I_t, \hat{c}_t) \le q_t^*(I_t, c_t) - d_t^*(I_t, c_t)$. Therefore, $q_t^*(I_t, \hat{c}_t) \le q_t^*(I_t, c_t)$. Thus, we have that $I_t^*(\hat{c}_t) \le I_t^*(c_t)$.

To conclude the proof, we need to show $\partial_{I_t} V_t(I_t|\hat{c}_t) - \hat{c}_t \le \partial_{I_t} V_t(I_t|c_t) - c_t$, for any $\hat{c}_t > c_t$. If $I_t \le x_t(c_t)$, $\partial_{I_t} V_t(I_t|\hat{c}_t) - \hat{c}_t \le 0 = \partial_{I_t} V_t(I_t|c_t) - c_t$. If $I_t \in [x_t(c_t), I_t^*(\hat{c}_t)]$ (without loss of generality, we assume $x_t(c_t) \le I_t^*(\hat{c}_t)$), (11) holds. Therefore,

$$\begin{aligned} \partial_{I_t} V_t(I_t|\hat{c}_t) - \hat{c}_t =& b - \hat{c}_t + \alpha\mu_t(\hat{c}_t) + \partial_y L(I_t - d_t^*(I_t, \hat{c}_t)) - \alpha\mu_t(\hat{c}_t) + \hat{c}_t \\ =& b + \partial_y L(I_t - d_t^*(I_t, \hat{c}_t)) \\ \le& b + \partial_y L(I_t - d_t^*(I_t, c_t)) \\ =& \partial_{I_t} V_t(I_t|c_t) - c_t, \end{aligned}$$

where the first equality follows from the envelope theorem and the inequality follows from the concavity of $L(\cdot)$. If $I_t \in [I_t^*(\hat{c}_t), I_t^*(c_t)]$,

$$\alpha\mu_t(\hat{c}_t) - \hat{c}_t + \partial_y H_t(I_t + q_t^*(I_t, \hat{c}_t) - d_t^*(I_t, \hat{c}_t)|\hat{c}_t) \le \alpha\mu_t(c_t) - c_t + \partial_y H_t(I_t + q_t^*(I_t, c_t) - d_t^*(I_t, c_t)|c_t) = 0.$$

Therefore,

$$\begin{aligned} \partial_{I_t} V_t(I_t|\hat{c}_t) - \hat{c}_t \le& b - \hat{c}_t + \alpha\mu_t(\hat{c}_t) + \partial_y L(I_t - d_t^*(I_t, \hat{c}_t)) - \alpha\mu_t(\hat{c}_t) + \hat{c}_t \\ =& b + \partial_y L(I_t - d_t^*(I_t, \hat{c}_t)) \\ \le& b + \partial_y L(I_t - d_t^*(I_t, c_t)) \\ =& \partial_{I_t} V_t(I_t|c_t) - c_t. \end{aligned}$$

If $I_t \ge I_t^*(c_t)$, $q_t^*(I_t, \hat{c}_t) = q_t^*(I_t, c_t) = 0$. Since $d_t^*(I_t, \hat{c}_t) \le d_t^*(I_t, c_t)$ and $\partial_y H_t(y|\hat{c}_t) \le \partial_y H_t(y|c_t)$,

$$\begin{aligned} \partial_{I_t} V_t(I_t|\hat{c}_t) - \hat{c}_t =& b - \hat{c}_t + \alpha\mu_t(\hat{c}_t) + \partial_y L(I_t - d_t^*(I_t, \hat{c}_t)) + \partial_y H_t(I_t - d_t^*(I_t, \hat{c}_t)|\hat{c}_t) \\ \le& b - c_t + \alpha\mu_t(c_t) + \partial_y L(I_t - d_t^*(I_t, c_t)) + \partial_y H_t(I_t - d_t^*(I_t, c_t)|c_t) \\ =& \partial_{I_t} V_t(I_t|c_t) - c_t. \end{aligned}$$

Thus, $\partial_{I_t} V_t(I_t|\hat{c}_t) - \hat{c}_t \le \partial_{I_t} V_t(I_t|c_t) - c_t$ for all I_t. *Q.E.D.*

Proof of Theorem 7: We show **parts (a) - (e)** together by backward induction.

Without loss of generality, we assume that $\hat{x}_t(c_t), x_t(c_t) > -\infty$. Our argument can be easily extended to the extreme case in which $\hat{x}_t(c_t) = -\infty$ or $x_t(c_t) = -\infty$. We only provide the proof for the case $q_t(c_t) > 0$, since the other case, $q_t(c_t)=0$, can be proved using the same method with simpler argument. Rewrite the objective function $J_t(x_t,q_t,d_t|c_t)$ as (2). Correspondingly, we define $\hat{J}_t(\cdot,\cdot,\cdot|c_t)$ and $\hat{\Psi}_t(\cdot|c_t)$ as the counterparts of $J_t(\cdot,\cdot,\cdot|c_t)$ and $\Psi_t(\cdot|c_t)$ in the model with procurement cost process $\{\hat{s}_t(c_t,\xi_t)\}_{t=T}^1$.

We first show that if $\partial_y\hat{\Psi}_t(y|c_t)\ge\partial_y\Psi_t(y|c_t)$ for all y, parts (b) - (e) hold. This condition holds for $t=t_*$. The first order condition with respect to x_t implies that

$$\begin{aligned}&\partial_y\Lambda(\hat{x}_t(c_t)-\hat{d}_t(c_t))+\partial_y\hat{\Psi}_t(\hat{x}_t(c_t)+\hat{q}_t(c_t)-\hat{d}_t(c_t)|c_t)\\=&\partial_y\Lambda(x_t(c_t)-d_t(c_t))+\partial_y\Psi_t(x_t(c_t)+q_t(c_t)-d_t(c_t)|c_t)=c_t.\end{aligned}\tag{12}$$

By (1), $\hat{d}_t(c_t)=d_t(c_t)$. Since $q_t(c_t)>0$, from (12), $\hat{\Delta}_t(c_t)=\Delta_t(c_t)$, $\hat{x}_t(c_t)=x_t(c_t)$, and $\hat{q}_t(c_t)\ge q_t(c_t)$.

If $I_t\ge x_t(c_t)=\hat{x}_t(c_t)$, assume that $\hat{q}_t^*(I_t,c_t)<q_t^*(I_t,c_t)$. Lemma 2 implies that $\partial_y\Psi_t(I_t+q_t^*(I_t,c_t)-d_t^*(I_t,c_t)|c_t)\ge\partial_y\hat{\Psi}_t(I_t+\hat{q}_t^*(I_t,c_t)-\hat{d}_t^*(I_t,c_t)|c_t)$. Since $\partial_y\hat{\Psi}_t(y|c_t)\ge\partial_y\Psi_t(y|c_t)$ for all y, $q_t^*(I_t,c_t)-d_t^*(I_t,c_t)\le\hat{q}_t^*(I_t,c_t)-\hat{d}_t^*(I_t,c_t)$. Hence, $d_t^*(I_t,c_t)>\hat{d}_t^*(I_t,c_t)$. Thus, $\Delta_t^*(I_t,c_t)=I_t-d_t^*(I_t,c_t)<I_t-\hat{d}_t^*(I_t,c_t)=\hat{\Delta}_t^*(I_t,c_t)$. By the concavity of $\Lambda(\cdot)$, we have: $\partial_y\Lambda(\Delta_t^*(I_t,c_t))+\partial_y\Psi_t^*(q_t^*(I_t,c_t)+\Delta_t^*(I_t,c_t)|c_t)\ge\partial_y\Lambda(\hat{\Delta}_t^*(I_t,c_t))+\partial_y\hat{\Psi}_t^*(\hat{q}_t^*(I_t,c_t)+\hat{\Delta}_t^*(I_t,c_t)|c_t)$. By Lemma 2, $d_t^*(I_t,c_t)>\hat{d}_t^*(I_t,c_t)$ implies that $\partial_{d_t}J_t(I_t,q_t^*(I_t,c_t),d_t^*(I_t,c_t)|c_t)\ge\partial_{d_t}\hat{J}_t(I_t,\hat{q}_t^*(I_t,c_t),\hat{d}_t^*(I_t,c_t)|c_t)$. Therefore,

$$\begin{aligned}R'(d_t^*(I_t,c_t))=&\partial_{d_t}J_t(I_t,q_t^*(I_t,c_t),d_t^*(I_t,c_t)|c_t)+\partial_y\Lambda(\Delta_t^*(I_t,c_t))+\partial_y\Psi_t(q_t^*(I_t,c_t)+\Delta_t^*(I_t,c_t)|c_t)\\\ge&\partial_{d_t}\hat{J}_t(I_t,\hat{q}_t^*(I_t,c_t),\hat{d}_t^*(I_t,c_t)|c_t)+\partial_y\Lambda(\hat{\Delta}_t^*(I_t,c_t))+\partial_y\hat{\Psi}_t(\hat{q}_t^*(I_t,c_t)+\hat{\Delta}_t^*(I_t,c_t)|c_t)\\=&R'(\hat{d}_t^*(I_t,c_t))\end{aligned}$$

However, $d_t^*(I_t,c_t)>\hat{d}_t^*(I_t,c_t)$ suggests that $R'(d_t^*(I_t,c_t))<R'(\hat{d}_t^*(I_t,c_t))$. This contradiction implies that $\hat{q}_t^*(I_t,c_t)\ge q_t^*(I_t,c_t)$ and, thus $\hat{I}_t^*(c_t)\ge I_t^*(c_t)$.

If $I_t\in[x_t^*(c_t),I_t^*(c_t)]$, $q_t^*(I_t,c_t)>0$ and $\hat{q}_t^*(I_t,c_t)>0$. The first order condition with respect to q_t implies that $\partial_y\Psi_t(I_t+q_t^*(I_t,c_t)-d_t^*(I_t,c_t))=\partial_y\hat{\Psi}_t(I_t+\hat{q}_t^*(I_t,c_t)-\hat{d}_t^*(I_t,c_t))=\gamma c_t$. This equality, together with the first order condition with respect to d_t, implies that $d_t^*(I_t,c_t)=\hat{d}_t^*(I_t,c_t)$. If $I_t\ge\hat{I}_t^*(c_t)$, $q_t^*(I_t,c_t)=\hat{q}_t^*(I_t,c_t)=0$. If $d_t^*(I_t,c_t)<\hat{d}_t^*(I_t,c_t)$, Lemma 2 implies that $\partial_{d_t}J_t(I_t,0,d_t^*(I_t,c_t)|c_t)\ge\partial_{d_t}\hat{J}_t(I_t,0,\hat{d}_t^*(I_t,c_t)|c_t)$. Therefore,

$$\begin{aligned}R'(d_t^*(I_t,c_t))=&\partial_{d_t}J_t(I_t,0,d_t^*(I_t,c_t)|c_t)+\partial_y\Lambda(\Delta_t^*(I_t,c_t))+\partial_y\Psi_t(\Delta_t^*(I_t,c_t)|c_t)\\\ge&\partial_{d_t}\hat{J}_t(I_t,0,\hat{d}_t^*(I_t,c_t)|c_t)+\partial_y\Lambda(\hat{\Delta}_t^*(I_t,c_t))+\partial_y\hat{\Psi}_t(\hat{\Delta}_t^*(I_t,c_t)|c_t)\\=&R'(\hat{d}_t^*(I_t,c_t))\end{aligned}$$

However, $d_t^*(I_t,c_t)>\hat{d}_t^*(I_t,c_t)$ suggests that $R'(d_t^*(I_t,c_t))<R'(\hat{d}_t^*(I_t,c_t))$. This contradiction implies that $d_t^*(I_t,c_t)\le\hat{d}_t^*(I_t,c_t)$ for $I_t\ge\hat{I}_t^*(c_t)$.

If $I_t^*(c_t)<I_t<\hat{I}_t^*(c_t)$, $\hat{q}_t^*(I_t,c_t)>0$ and $q_t^*(I_t,c_t)=0$. The first order condition with respect to q_t implies that $\partial_y\Psi_t(I_t-d_t^*(I_t,c_t))\le\partial_y\hat{\Psi}_t(I_t+\hat{q}_t^*(I_t,c_t)-\hat{d}_t^*(I_t,c_t))=\gamma c_t$. If $d_t^*(I_t,c_t)<\hat{d}_t^*(I_t,c_t)$, Lemma 2 yields that $\partial_{d_t}\hat{J}_t(I_t,\hat{q}_t^*(I_t,c_t),\hat{d}_t^*(I_t,c_t))\ge\partial_{d_t}J_t(I_t,0,d_t^*(I_t,c_t))$. Therefore,

$$\begin{aligned}R'(d_t^*(I_t,c_t))=&\partial_{d_t}J_t(I_t,0,d_t^*(I_t,c_t)|c_t)+\partial_y\Lambda(I_t-d_t^*(I_t,c_t))+\partial_y\Psi_t(I_t-d_t^*(I_t,c_t)|c_t)\\\le&\partial_{d_t}\hat{J}_t(I_t,\hat{q}_t^*(I_t,c_t),\hat{d}_t^*(I_t,c_t)|c_t)+\partial_y\Lambda(I_t-\hat{d}_t^*(I_t,c_t))+\partial_y\hat{\Psi}_t(I_t+\hat{q}_t^*(I_t,c_t)-\hat{d}_t^*(I_t,c_t)|c_t)\\=&R'(\hat{d}_t^*(I_t,c_t))\end{aligned}$$

However, $d_t^*(I_t,c_t)<\hat{d}_t^*(I_t,c_t)$ implies that $R'(d_t^*(I_t,c_t))>R'(\hat{d}_t^*(I_t,c_t))$. This contradiction shows that $d_t^*(I_t,c_t)\ge\hat{d}_t^*(I_t,c_t)$ for all $I_t\ge x_t^*(c_t)$ if $\partial_y\hat{\Psi}_t(y|c_t)\ge\partial_y\Psi_t(y|c_t)$ for all y. $\hat{\Delta}_t^*(I_t,c_t)\ge\Delta_t^*(I_t,c_t)$ then follows from $d_t^*(I_t,c_t)\ge\hat{d}_t^*(I_t,c_t)$.

To complete the induction, we show that if $\partial_y\hat{\Psi}_t(y|c_t)\geq\partial_y\Psi_t(y|c_t)$ for all y, $\partial_{I_t}\hat{V}_t(I_t|c_t)\geq\partial_{I_t}V_t(I_t|c_t)$ for all I_t. If $I_t\leq x_t(c_t)=\hat{x}_t(c_t)$, $\partial_{I_t}\hat{V}_t(I_t|c_t)=\partial_{I_t}V_t(I_t|c_t)=c_t$.

If $I_t\in[x_t(c_t),I_t^*(c_t)]$, $\partial_y\Psi_t(I_t+q_t^*(I_t,c_t)-d_t^*(I_t,c_t)|c_t)=\partial_y\hat{\Psi}_t(I_t+\hat{q}_t^*(I_t,c_t)-\hat{d}_t^*(I_t,c_t)|c_t)=\gamma c_t$, and $d_t^*(I_t,c_t)=\hat{d}_t^*(I_t,c_t)$. Hence,

$$\begin{aligned}\partial_{I_t}\hat{V}_t(I_t|c_t)=&\partial_y\Lambda(I_t-\hat{d}_t^*(I_t,c_t))+\partial_y\hat{\Psi}_t(I_t+\hat{q}_t^*(I_t,c_t)-\hat{d}_t^*(I_t,c_t)|c_t)\\=&\partial_y\Lambda(I_t-d_t^*(I_t,c_t))+\partial_y\Psi_t(I_t+q_t^*(I_t,c_t)-d_t^*(I_t,c_t)|c_t)\\=&\partial_{I_t}V_t(I_t|c_t).\end{aligned}$$

If $I_t\in[I_t^*(c_t),\hat{I}_t^*(c_t)]$, $\partial_y\Psi_t(I_t-d_t^*(I_t,c_t)|c_t)\leq\partial_y\hat{\Psi}_t(I_t+\hat{q}_t^*(I_t,c_t)-\hat{d}_t^*(I_t,c_t)|c_t)=\gamma c_t$, and $d_t^*(I_t,c_t)\geq\hat{d}_t^*(I_t,c_t)$. We consider two cases $d_t^*(I_t,c_t)=\hat{d}_t^*(I_t,c_t)$ and $d_t^*(I_t,c_t)>\hat{d}_t^*(I_t,c_t)$. The same argument as the one in the proof of Theorem 4 yields that $\partial_{I_t}\hat{V}_t(I_t|c_t)\geq\partial_{I_t}V_t(I_t|c_t)$ for $I_t\in[I_t^*(c_t),\hat{I}_t^*(c_t)]$.

If $I_t\geq\hat{I}_t^*(c_t)$, $q_t^*(I_t,c_t)=\hat{q}_t^*(I_t,c_t)=0$. The same argument as the one in the proof of Theorem 4 shows that $\partial_{I_t}\hat{V}_t(I_t|c_t)\geq\partial_{I_t}V_t(I_t|c_t)$ for $I_t\geq\hat{I}_t^*(c_t)$. Finally, $\partial_{I_t}\hat{V}_t(I_t|c_t)\geq\partial_{I_t}V_t(I_t|c_t)$ yields that

$$\partial_y\hat{\Psi}_{t+1}(y|c_{t+1})=\alpha\mathbb{E}\{\partial_{I_t}\hat{V}_t(y-\epsilon_t|s_{t+1}(c_{t+1},\xi_{t+1}))\}\geq\alpha\mathbb{E}\{\partial_{I_t}V_t(y-\epsilon_t|s_{t+1}(c_{t+1},\xi_{t+1}))\}=\partial_y\Psi_{t+1}(y|c_{t+1}).$$

Since $\partial_y\hat{\Psi}_t(y|c_t)\geq\partial_y\Psi_t(y|c_t)$ for $t=t_*$. The initial condition is satisfied. Hence, Theorem 7 follows for all $t\geq t_*$. *Q.E.D.*

Proof of Theorem 8: We prove **parts (a) - (c)** together by backward induction.

Without loss of generality, we assume that $x_{\hat{\gamma},t}(c_t),x_{\gamma,t}(c_t)>-\infty$ and $q_{\hat{\gamma},t}(c_t),q_{\gamma,t}(c_t)>0$. Our argument can be easily extended to the extreme case in which $x_{\hat{\gamma},t}(c_t)=-\infty$, $x_{\gamma,t}(c_t)=-\infty$, $q_{\hat{\gamma},t}(c_t)=0$, or $q_{\gamma,t}(c_t)=0$. We need to show that if $\partial_{I_{t-1}}V_{\hat{\gamma},t-1}(I_{t-1}|c_{t-1})\geq\partial_{I_{t-1}}V_{\gamma,t-1}(I_{t-1}|c_{t-1})$, (a) $\Delta_{\hat{\gamma},t}(c_t)\geq\Delta_{\gamma,t}(c_t)$, (b) $d^*_{\hat{\gamma},t}(I_t,c_t)\leq d^*_{\gamma,t}(I_t,c_t)$, and (c) $\partial_{I_t}V_{\hat{\gamma},t}(I_t|c_t)\geq\partial_{I_t}V_{\gamma,t}(I_t|c_t)$. Since $V_{\hat{\gamma},0}(\cdot|c_0)=V_{\gamma,0}(\cdot|c_0)\equiv0$, the initial condition is satisfied.

We define $\Psi_{\hat{\gamma},t}(y|c_t):=\mathbb{E}\{V_{\hat{\gamma},t}(y-\epsilon_t|s_t(c_t,\xi_t))|c_t\}$ and $\Psi_{\gamma,t}(y|c_t):=\mathbb{E}\{V_{\gamma,t}(y-\epsilon_t|s_t(c_t,\xi_t))|c_t\}$. It's clear from $\partial_{I_{t-1}}V_{\hat{\gamma},t-1}(I_{t-1}|c_{t-1})\geq\partial_{I_{t-1}}V_{\gamma,t-1}(I_{t-1}|c_{t-1})$ that $\partial_y\Psi_{\hat{\gamma},t}(y|c_t)\geq\partial_y\Psi_{\gamma,t}(y|c_t)$ for any y. $d_{\hat{\gamma},t}(c_t)=d_{\gamma,t}(c_t)$ follows directly from equation (1). The first-order condition with respect to q_t implies that $\partial_y\Psi_{\hat{\gamma},t}(\Delta_{\hat{\gamma},t}(c_t)+q_{\hat{\gamma},t}(c_t)|c_t)=\hat{\gamma}c_t>\gamma c_t=\partial_y\Psi_{\gamma,t}(\Delta_{\gamma,t}(c_t)+q_{\gamma,t}(c_t)|c_t)$, and that with respect to x_t implies that $\partial_y\Lambda(\Delta_{\hat{\gamma},t}(c_t))+\partial_y\Psi_{\hat{\gamma},t}(\Delta_{\hat{\gamma},t}(c_t)+q_{\hat{\gamma},t}(c_t)|c_t)=\partial_y\Lambda(\Delta_{\gamma,t}(c_t))+\partial_y\Psi_{\gamma,t}(\Delta_{\gamma,t}(c_t)+q_{\gamma,t}(c_t)|c_t)=c_t$. Hence, $\partial_y\Lambda(\Delta_{\hat{\gamma},t}(c_t))=(1-\hat{\gamma})c_t<(1-\gamma)c_t=\partial_y\Lambda(\Delta_{\gamma,t}(c_t))$. The concavity of $\Lambda(\cdot)$ yields that $\Delta_{\hat{\gamma},t}(c_t)\geq\Delta_{\gamma,t}(c_t)$ and, thus, $x_{\hat{\gamma},t}(c_t)=\Delta_{\hat{\gamma},t}(c_t)+d_{\hat{\gamma},t}(c_t)\geq\Delta_{\gamma,t}(c_t)+d_{\gamma,t}(c_t)=x_{\gamma,t}(c_t)$. It follows immediately that $\Delta^*_{\hat{\gamma},t}(I_t,c_t)\geq\Delta^*_{\gamma,t}(I_t,c_t)$ and $x^*_{\hat{\gamma},t}(I_t,c_t)\geq x^*_{\gamma,t}(I_t,c_t)$.

If $I_t\in[x_{\gamma,t}(c_t),x_{\hat{\gamma},t}(c_t)]$, $d^*_{\hat{\gamma},t}(I_t,c_t)=d_{\hat{\gamma},t}(c_t)=d_{\gamma,t}(c_t)\leq d^*_{\gamma,t}(I_t,c_t)$. If $I_t\geq x_{\hat{\gamma},t}(c_t)$ and $d^*_{\hat{\gamma},t}(I_t,c_t)>d^*_{\gamma,t}(I_t,c_t)$, $\partial_y\Lambda(I_t-d^*_{\hat{\gamma},t}(I_t,c_t))\geq\partial_y\Lambda(I_t-d^*_{\gamma,t}(I_t,c_t))$. There are two cases: (a) $q^*_{\hat{\gamma},t}(I_t,c_t)>q^*_{\gamma,t}(I_t,c_t)$, and (b)$q^*_{\hat{\gamma},t}(I_t,c_t)\leq q^*_{\gamma,t}(I_t,c_t)$. If $q^*_{\hat{\gamma},t}(I_t,c_t)>q^*_{\gamma,t}(I_t,c_t)$, Lemma 2 implies that $\partial_y\Psi_{\hat{\gamma},t}(I_t+q^*_{\hat{\gamma},t}(I_t,c_t)-d^*_{\hat{\gamma},t}(I_t,c_t)|c_t)\geq\partial_y\Psi_{\gamma,t}(I_t+q^*_{\gamma,t}(I_t,c_t)-d^*_{\gamma,t}(I_t,c_t)|c_t)$. If $q^*_{\hat{\gamma},t}(I_t,c_t)\leq q^*_{\gamma,t}(I_t,c_t)$, $\partial_y\Psi_{\hat{\gamma},t}(y|c_t)\geq\partial_y\Psi_{\gamma,t}(y|c_t)$ and the concavity of $\Psi_{\hat{\gamma},t}(\cdot|c_t)$ and $\Psi_{\gamma,t}(\cdot|c_t)$ imply that $\partial_y\Psi_{\hat{\gamma},t}(I_t+q^*_{\hat{\gamma},t}(I_t,c_t)-d^*_{\hat{\gamma},t}(I_t,c_t)|c_t)\geq\partial_y\Psi_{\gamma,t}(I_t+$

$q^*_{\gamma,t}(I_t,c_t)-d^*_{\gamma,t}(I_t,c_t)|c_t)$. Thus, in both cases, $\partial_y\Psi_{\hat\gamma,t}(I_t+q^*_{\hat\gamma,t}(I_t,c_t)-d^*_{\hat\gamma,t}(I_t,c_t)|c_t)\ge\partial_y\Psi_{\gamma,t}(I_t+q^*_{\gamma,t}(I_t,c_t)-d^*_{\gamma,t}(I_t,c_t)|c_t)$. Since $d^*_{\hat\gamma,t}(I_t,c_t)>d^*_{\gamma,t}(I_t,c_t)$, Lemma 2 implies that $\partial_{d_t}J_{\hat\gamma,t}(I_t,q^*_{\hat\gamma,t}(I_t,c_t),d^*_{\hat\gamma,t}(I_t,c_t)|c_t)\ge\partial_{d_t}J_{\gamma,t}(I_t,q^*_{\gamma,t}(I_t,c_t),d^*_{\gamma,t}(I_t,c_t)|c_t)$. Therefore, we have:

$$\begin{aligned}R'(d^*_{\hat\gamma,t}(I_t,c_t))=&\partial_{d_t}J_{\hat\gamma,t}(I_t,q^*_{\hat\gamma,t}(I_t,c_t),d^*_{\hat\gamma,t}(I_t,c_t)|c_t)+\partial_y\Lambda(I_t-d^*_{\hat\gamma,t}(I_t,c_t))+\partial_y\Psi_{\hat\gamma,t}(I_t+q^*_{\hat\gamma,t}(I_t,c_t)-d^*_{\hat\gamma,t}(I_t,c_t)|c_t)\\ \ge&\partial_{d_t}J_{\gamma,t}(I_t,q^*_{\gamma,t}(I_t,c_t),d^*_{\gamma,t}(I_t,c_t)|c_t)+\partial_y\Lambda(I_t-d^*_{\gamma,t}(I_t,c_t))+\partial_y\Psi_{\gamma,t}(I_t+q^*_{\gamma,t}(I_t,c_t)-d^*_{\gamma,t}(I_t,c_t)|c_t)\\ =&R'(d^*_{\gamma,t}(I_t,c_t)),\end{aligned}$$

which contradicts the strict concavity of $R(\cdot)$. Hence, $d^*_{\hat\gamma,t}(I_t,c_t)\le d^*_{\gamma,t}(I_t,c_t)$ for all I_t.

To complete the induction, it suffices to show that $\partial_{I_t}V_{\hat\gamma,t}(I_t|c_t)\ge\partial_{I_t}V_{\gamma,t}(I_t|c_t)$ for all I_t. If $I_t\le x_{\hat\gamma,t}(c_t)$, $\partial_{I_t}V_{\hat\gamma,t}(I_t|c_t)=c_t\ge\partial_{I_t}V_{\gamma,t}(I_t|c_t)$. Now we consider the case $I_t>x_{\hat\gamma,t}(c_t)$. There are two cases (a) $d^*_{\hat\gamma,t}(I_t,c_t)=d^*_{\gamma,t}(I_t,c_t)$, and (b) $d^*_{\hat\gamma,t}(I_t,c_t)<d^*_{\gamma,t}(I_t,c_t)$.

If $d^*_{\hat\gamma,t}(I_t,c_t)=d^*_{\gamma,t}(I_t,c_t)$, $\partial_y\Lambda(I_t-d^*_{\hat\gamma,t}(I_t,c_t))=\partial_y\Lambda(I_t-d^*_{\gamma,t}(I_t,c_t))$. Moreover, if $q^*_{\hat\gamma,t}(I_t,c_t)>q^*_{\gamma,t}(I_t,c_t)$, Lemma 2 implies that $\partial_y\Psi_{\hat\gamma,t}(I_t+q^*_{\hat\gamma,t}(I_t,c_t)-d^*_{\hat\gamma,t}(I_t,c_t)|c_t)\ge\partial_y\Psi_{\gamma,t}(I_t+q^*_{\gamma,t}(I_t,c_t)-d^*_{\gamma,t}(I_t,c_t)|c_t)$. If $q^*_{\hat\gamma,t}(I_t,c_t)\le q^*_{\gamma,t}(I_t,c_t)$, $\partial_y\Psi_{\hat\gamma,t}(y|c_t)\ge\partial_y\Psi_{\gamma,t}(y|c_t)$ and the concavity of $\Psi_{\hat\gamma,t}(\cdot|c_t)$ and $\Psi_{\gamma,t}(\cdot|c_t)$ imply that $\partial_y\Psi_{\hat\gamma,t}(I_t+q^*_{\hat\gamma,t}(I_t,c_t)-d^*_{\hat\gamma,t}(I_t,c_t)|c_t)\ge\partial_y\Psi_{\gamma,t}(I_t+q^*_{\gamma,t}(I_t,c_t)-d^*_{\gamma,t}(I_t,c_t)|c_t)$. Thus, in both cases, $\partial_y\Psi_{\hat\gamma,t}(I_t+q^*_{\hat\gamma,t}(I_t,c_t)-d^*_{\hat\gamma,t}(I_t,c_t)|c_t)\ge\partial_y\Psi_{\gamma,t}(I_t+q^*_{\gamma,t}(I_t,c_t)-d^*_{\gamma,t}(I_t,c_t)|c_t)$. Hence, $\partial_{I_t}V_{\hat\gamma,t}(I_t|c_t)=\partial_y\Lambda(I_t-d^*_{\hat\gamma,t}(I_t,c_t))+\partial_y\Psi_{\hat\gamma,t}(I_t+q^*_{\hat\gamma,t}(I_t,c_t)-d^*_{\hat\gamma,t}(I_t,c_t)|c_t)\ge\partial_y\Lambda(I_t-d^*_{\gamma,t}(I_t,c_t))+\partial_y\Psi_{\gamma,t}(I_t+q^*_{\gamma,t}(I_t,c_t)-d^*_{\gamma,t}(I_t,c_t)|c_t)=\partial_{I_t}V_{\gamma,t}(I_t|c_t)$.

If $d^*_{\hat\gamma,t}(I_t,c_t)<d^*_{\gamma,t}(I_t,c_t)$, $R'(d^*_{\hat\gamma,t}(I_t,c_t))>R'(d^*_{\gamma,t}(I_t,c_t))$, and, by Lemma 2, $\partial_{d_t}J_{\hat\gamma,t}(I_t,q^*_{\hat\gamma,t}(I_t,c_t),d^*_{\hat\gamma,t}(I_t,c_t)|c_t)\le\partial_{d_t}J_{\gamma,t}(I_t,q^*_{\gamma,t}(I_t,c_t),d^*_{\gamma,t}(I_t,c_t)|c_t)$. Therefore,

$$\begin{aligned}\partial_{I_t}V_{\hat\gamma,t}(I_t|c_t)=&\partial_y\Lambda(I_t-d^*_{\hat\gamma,t}(I_t,c_t))+\partial_y\Psi_{\hat\gamma,t}(I_t+q^*_{\hat\gamma,t}(I_t,c_t)-d^*_{\hat\gamma,t}(I_t,c_t)|c_t)\\ =&R'(d^*_{\hat\gamma,t}(I_t,c_t))-\partial_{d_t}J_{\hat\gamma,t}(I_t,q^*_{\hat\gamma,t}(I_t,c_t),d^*_{\hat\gamma,t}(I_t,c_t)|c_t)\\ \ge&R'(d^*_{\gamma,t}(I_t,c_t))-\partial_{d_t}J_{\gamma,t}(I_t,q^*_{\gamma,t}(I_t,c_t),d^*_{\gamma,t}(I_t,c_t)|c_t)\\ =&\partial_y\Lambda(I_t-d^*_{\gamma,t}(I_t,c_t))+\partial_y\Psi_{\gamma,t}(I_t+q^*_{\gamma,t}(I_t,c_t)-d^*_{\gamma,t}(I_t,c_t)|c_t)\\ =&\partial_{I_t}V_{\gamma,t}(I_t|c_t).\end{aligned}$$

Therefore, $\partial_{I_t}V_{\hat\gamma,t}(I_t|c_t)\ge\partial_{I_t}V_{\gamma,t}(I_t|c_t)$ for all I_t. This completes the induction and, thus, the proof of **parts (a) - (c)**.

Part (d) follows from analogous argument to the proof of Theorem 6. Hence, we omit its proof for brevity. *Q.E.D.*

Numerical Studies

We now specify the transition probability matrices for the procurement cost processes in Sections 6.2 - 6.3, and give a numerical example in which the optimal forward-buying quantity is not monotone in the current procurement cost.

Transition Probability Matrix in Section 6.2

The transition probability matrix for the procurement cost process in Section 6.2, P, is given by:

$$P_{ij} = \begin{cases} 1/6, & \text{if } i = 0, 20,\ |j-i| \le 5; \\ 1/7, & \text{if } i = 1, 19,\ |j-i| \le 5; \\ 1/8, & \text{if } i = 2, 18,\ |j-i| \le 5; \\ 1/9, & \text{if } i = 3, 17,\ |j-i| \le 5; \\ 1/10, & \text{if } i = 4, 16,\ |j-i| \le 5; \\ 1/11, & \text{if } 5 \le i \le 15,\ |j-i| \le 5; \\ 0, & \text{otherwise.} \end{cases}$$

Transition Probability Matrices in Section 6.3

We use P, $\hat{P}$, and $\hat{\hat{P}}$ to denote the transition probability matrix for $\{c_t\}$, $\{\hat{c}_t\}$, and $\{\hat{\hat{c}}_t\}$, respectively. Let P_i, $\hat{P}_i$, and $\hat{\hat{P}}_i$ denote the i^{th} row vector of P, $\hat{P}$, and $\hat{\hat{P}}$.

For $i = 0, 1, 2$,

$$\begin{cases} P_i = (\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, 0,0,0,0,0,0,0,0,0,0,0,0,0,0), \\ \hat{P}_i = (\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, 0,0,0,0,0,0,0,0,0,0,0,0,0,0), \\ \hat{\hat{P}}_i = (\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, 0,0,0,0,0,0,0,0,0,0,0,0,0,0). \end{cases}$$

For $i = 3$,

$$\begin{cases} P_i = (0, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, 0,0,0,0,0,0,0,0,0,0,0,0,0), \\ \hat{P}_i = (\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0,0,0,0,0,0,0,0,0,0,0,0), \\ \hat{\hat{P}}_i = (\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0,0,0,0,0,0,0,0,0,0,0,0). \end{cases}$$

For $i = 4$,

$$\begin{cases} P_i = (0, 0, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, 0,0,0,0,0,0,0,0,0,0,0,0), \\ \hat{P}_i = (0, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0,0,0,0,0,0,0,0,0,0,0), \\ \hat{\hat{P}}_i = (\frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, 0,0,0,0,0,0,0,0,0,0). \end{cases}$$

For $i = 5$,

$$\begin{cases} P_i = (0, 0, 0, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, 0,0,0,0,0,0,0,0,0,0,0), \\ \hat{P}_i = (0, 0, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0,0,0,0,0,0,0,0,0,0), \\ \hat{\hat{P}}_i = (0, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, 0,0,0,0,0,0,0,0,0). \end{cases}$$

For $i = 6, 7, 8$,

$$\begin{cases} P_i = (0, 0, 0, 0, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, 0,0,0,0,0,0,0,0,0,0), \\ \hat{P}_i = (0, 0, 0, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0,0,0,0,0,0,0,0,0), \\ \hat{\hat{P}}_i = (0, 0, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, 0,0,0,0,0,0,0,0). \end{cases}$$

For $i = 9$,

$$\begin{cases} P_i = (0,0,0,0,0,0,0, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, 0,0,0,0,0,0,0), \\ \hat{P}_i = (0,0,0,0,0,0, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0,0,0,0,0,0), \\ \hat{\hat{P}}_i = (0,0,0,0,0, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, 0,0,0,0,0). \end{cases}$$

For $i = 10$,

$$\begin{cases} P_i = (0,0,0,0,0,0,0,0, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, 0,0,0,0,0,0), \\ \hat{P}_i = (0,0,0,0,0,0,0, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0,0,0,0,0), \\ \hat{\hat{P}}_i = (0,0,0,0,0,0, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, 0,0,0,0). \end{cases}$$

For $i = 11$,

$$\begin{cases} P_i = (0,0,0,0,0,0,0,0,0, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, 0,0,0,0,0), \\ \hat{P}_i = (0,0,0,0,0,0,0,0, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0,0,0,0), \\ \hat{\hat{P}}_i = (0,0,0,0,0,0,0, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, 0,0,0). \end{cases}$$

For $i=12,13,14$,

$$\begin{cases} P_i=(0,0,0,0,0,0,0,0,0,0,\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},0,0,0,0), \\ \hat{P}_i=(0,0,0,0,0,0,0,0,0,\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},0,0,0), \\ \hat{\hat{P}}_i=(0,0,0,0,0,0,0,0,\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},0,0). \end{cases}$$

For $i=15$,

$$\begin{cases} P_i=(0,0,0,0,0,0,0,0,0,0,0,\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},0,0,0), \\ \hat{P}_i=(0,0,0,0,0,0,0,0,0,0,\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},0,0), \\ \hat{\hat{P}}_i=(0,0,0,0,0,0,0,0,0,\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},0). \end{cases}$$

For $i=16$,

$$\begin{cases} P_i=(0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},0,0), \\ \hat{P}_i=(0,0,0,0,0,0,0,0,0,0,0,\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},0), \\ \hat{\hat{P}}_i=(0,0,0,0,0,0,0,0,0,0,\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11}). \end{cases}$$

For $i=17$,

$$\begin{cases} P_i=(0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},0), \\ \hat{P}_i=(0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9}), \\ \hat{\hat{P}}_i=(0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9}). \end{cases}$$

For $i=18,19,20$,

$$\begin{cases} P_i=(0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7}), \\ \hat{P}_i=(0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7}), \\ \hat{\hat{P}}_i=(0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7}). \end{cases}$$

It's clear from the entries of P, $\hat{P}$, and $\hat{\hat{P}}$ that $\hat{\hat{s}}_t(c_t,\xi_t)\geq_{cx}\hat{s}_t(c_t,\xi_t)\geq_{cx}s_t(c_t,\xi_t)$ for each c_t.

Non-Monotone Forward-Buying Quantities

In Theorem 6(c), we show that when the procurement cost grows more rapidly at a lower cost level (i.e., $\kappa_t(c_t)=\alpha\mu_t(c_t)-c_t$ is decreasing in c_t) and the spot-purchasing and forward-buying channels are equally costly (i.e., $\gamma=1$), the firm should order less through the forward-buying contract at a higher spot-purchasing cost. In this subsection, we give a numerical example to illustrate that when the above conditions are violated, there is no monotone relation between the optimal forward-buying quantities and the current procurement cost. We use the same numerical setup as in Section 6.1, except that the backlogging and holding costs, and spot market procurement cost processes are different. More specifically, in this example, the expected demand is linear in price: $d(p_t)=a-kp_t$ with market size $a=1$ and price sensitivity $k=1$. The random component of D_t follows *i.i.d.* normal distributions with mean 0 and standard deviation $\sigma=0.2$. The maximum expected demand is $\bar{d}=0.8$ and the minimum expected demand is $\underline{d}=0.2$. The holding cost is $h=0.05$ and the backlogging cost is $b=0.5$. We set $\alpha=0.99$ and $\gamma=0.95$. The planning horizon length is $T=2$. We assume that procurement cost is driven by a 5-state Markov chain given in Table 3.

We use P to denote the transition probability matrix of the cost process, where P_{ij} is the probability that the cost in the current period is c_j given that the cost in the previous period is c_i. P_{ij} can be summarized as follows:

$$P_{ij}=\begin{cases}1/2, & \text{if } i=0,4,\ |j-i|\leq 1;\\ 1/3, & \text{if } i=1,2,3,\ |j-i|\leq 1;\\ 0, & \text{otherwise.}\end{cases}$$

Table 3 Procurement Cost States

t	c_0	c_1	c_2	c_3	c_4
1	0.15	0.35	0.40	0.60	1.00
2	0.20	0.30	0.40	0.50	0.60

Given the above model setup, it's easy to see that $\kappa_t(c_t)$ is not decreasing in c_t for $t = 2$. Figure 3 plots the optimal forward-buying quantity for each procurement cost in period 2, $q_t(c_t)$, which is not monotone in the current spot market procurement cost c_t.

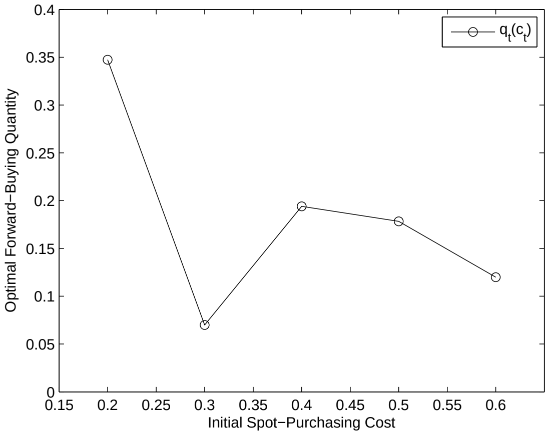

Figure 3 Optimal Forward-Buying Quantity

References

Durrett, R. 2010. *Probability, Theory and Examples, 4th ed.* (Cambirdge University Press, New York).